Analysis of Cost Functions for Markovian Arrival Queues

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***Abstract*-Cost analyses of Markovian arrival queues are considered. The arriving units are served by the behaviours of customers and servers. Bulk service, vacation, breakdown, repairs, are employed in the service systems. Mishra and Dinesh Kumar Yadav (2008) have analysed revenue, cost and profit analysis of a queueing system with Erlangian service queue. Jain et al (2011) have studied the total expected cost in different situations. The cost equations are derived for different queues by using their operating characteristics. By assuming the parameters, costs are computed and compared.**

***Keywords*-Markovian arrival, Erlangian service, vacation, breakdown, bulk service, mean queue length and cost function.**

I. Introduction

Researches on queueing theory have established several models since 1909. The models have been re-designed by using some concepts like bulk size rule, priority rule, feedback rule, retrial policy, vacation policy etc. They have derived explicit expressions and obtained numerical values. Recently the cost structure has been discussed in queueing systems. Few researchers have made some contributions on cost analysis of queues.

Mishra and Dinesh Kumar Yadav (2008) have analysed revenue, cost and profit analysis of a queueing system with Erlangian service queue. Jain et al (2011) have studied the total expected cost in different situations.

In this paper, an attempt is made to obtain the cost functions explicitly and numerically for various queueing models.

At first, consider singe and batch services is an Erlangian service with Poisson arrival queue. Based on the models, the cost functions are established. The costs and their curves are exhibited. Secondly, the cost function is derived for $M^x|G|1$queue with multiple vacations. The costs are obtained based on single and multiple vacations. The corresponding curves are drawn. Finally, consider a queue with vacation rule, service interruption and repairs. By using some defined probabilities for vacation, service interruption, repairs etc and mean queue length, total cost function is derived. The computed costs are obtained and the relavent curves are exhibited.

II. Cost Analysis of various queueing systems

A. $M|E_k|1$ queue with removable service

The cost analysis of queueing system with removable service station is very useful to provide basic framework for efficient design and analysis of several practical situations including various technical systems. For any queueing system cost and profit analysis constitutes a very important aspect of its investigation.

Consider an $M|E_k|1$ queueing system with a removable service station in which idle fraction of servers time is optimized by shifting the server to another service station in the turned off state of the system. It is assumed that customer arrival follows Poisson process with parameter λ and with service time according to an Erlang distribution with mean $\frac{1}{\mu}$ and stage parameter k. The Erlang type k distribution is made up of k independent and identical exponential stages; each with mean $\frac{1}{k\mu}$ The service operates when N customers have accumulated and is shutdown when no customers are present.

Wang and Huang (1995) have derived the expected queue lengths when the service station is turned off and that is turned on for $M|E_k|1$ queueing system are obtained as

$$L_{off} = \frac{(N-1)(1-\rho)}{2} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

$$L_{on} = \frac{\rho\left(N + 1 - \rho N + \frac{\rho}{k}\right)}{2(1-\rho)} \qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (2)$$

Now, we construct total expected cost as

$$T_C = C_s k\mu + C_0\, L_{off} + C_n\, L_{on} \qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (3)$$

where C_s = service cost per unit time

C_0 = holding cost for each customer per unit time when system is turned off.

C_n = holding cost per unit time when system is turned on,

Substituting the equations (1) and (2) in the cost equation (3). and providing suitable assumptions in the parameters of the equation (3) which yields.

$$T_c = 283.5 - 67.5\,\rho + \frac{1}{(1-\rho)}\,(66\rho - 58\rho^2) \qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (4)$$

By taking $\rho = 0.1, 0.2, \dots \dots, 0.9,$ in the equation (4) which gives the total expected costs (Table I) and its relevant curve is exhibited in Fig. 1.

TABLE I

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
T_c	283.4389	283.6	284.0786	285.0333	286.75	289.8	295.5167	307.9	346.95

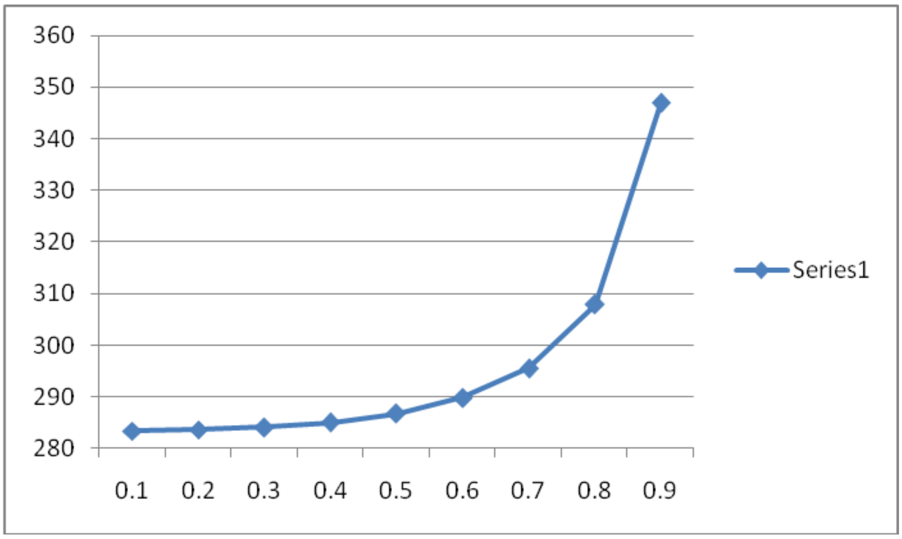

Fig. 1

B. $M/E_k^{(1,\alpha)}/1$ *queue with removable service*

A single server bulk service queuing system with removable service station is considered. The arrival of customers follows Poisson process with parameter λ. The arriving customer are served in batches of varying size $(1, \alpha)$ under usual bulk size rule which was designed by Bailey (1954). The service times of each batch are distributed according to an Erlang distribution with shape parameter k and scale parameter μ. The services are performed in k independent and identical exponential stages, each with mean $1/_{k\mu}$. An operator of the system can turn a single service station on at the time of customers arrival or off at the time of service completion. If the service station is in operation, then the arriving customers must wait until the station is available. For the above model, Joshua Joseph and Ganesan (2010) have derived the results as

$$L_{(off)} = \frac{N(N-1)}{2}\;\frac{(\alpha + k(\beta - r))}{\{N(\alpha + k + kr) + kr\,\alpha\}} \qquad \_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (5)$$

$$L_{on} = \frac{k}{4(2\propto - - kr)^2}\Big[r P_{0,0}^{0}[2(\propto - kr)\{(k-1)(k-2)r^2 - 3N(k-1)r + 3N(N-1)\} - 3\{2N-(k-1)r\}\{k(k-1)r^2 - \propto(\propto+1)\}] + [-2[(k-1)(k-2)\eta r^2 - 3(k-1)(\beta - \propto\eta)r + 3\{(\xi - (2\propto+1)\beta + \propto(\propto+1)\eta\}]] + 3\{2(\beta - \propto\eta) - (k-1)\eta r\}\{k(k-1)\eta^2 - \propto(\propto+1)\}]\Big] \_\_\_\_\_\_ (6)$$

$$L_N = \frac{1}{4(\propto - kr)^2\{n(\propto + kr) + kr\propto\}}[\{2N(N-1)(\propto - kr)^2 + kr[2(\propto - kr)(k-1)(k-2)r^2 - 3N(k-1)r + 3N(N-1)]\} - 3\{2N-(k-1)r\}\{k(k-1)r^2 - \propto(\propto+1)\}]\{\propto + k(\beta - r)\} + \{k\{N(\propto + kr) + kr \propto\}[-2(\propto - kr)\{(k-1)(k-2)\eta r^2 - 3(k-1)(\beta - \propto\eta)r + 3\{\xi - (2\propto+1)\beta + \propto(\propto+1)\eta)\}\} + 3\{2(\beta - \propto\eta) - (k-1)\eta r\}k(k-1)r^2 \propto (\propto+1)]\}] \_\_\_\_\_\_ (7)$$

$$E(I) = {}^{N}/_{\lambda} \qquad \_\_\_\_\_\_\_\_\_\_\_\_ \quad (8)$$

$$E(C) = \frac{N(\propto + kr) + kra}{\lambda\{\propto + k(\beta - r)\}} \qquad \_\_\_\_\_\_\_\_\_\_\_\_ \quad (9)$$

The construction of the cost structure is associated with various costs which are defined here.

C_b : holding cost per unit time for each customer present in the system.

C_s : Setup cost per unit time for activating the service station is turned off (on) removed from the system.

C_r : Removable cost per unit time for removing the service station. The removable cost is incurred each time the operating service station is removed from the service system and

C_a : Cost per unit time for performing an auxiliary task by the service station.

Based on the above cost constraints, the total expected cost function per unit time is defined as

$$T_c(N) = C_n L_n + C_0\left[\frac{E(B)}{E(C)}\right] + (C_s + C_r)\left[\frac{1}{E(C)}\right] + C_0\left[\frac{E(I)}{E(C)}\right] \qquad \_\_\_\_\_\_\_\_\_\_\_\_ \quad (10)$$

By using the equations (7), (8) and (9) in the equation (10) and apply some numerical for unknowns and we get the cost equation as

$$T_c(N) = \frac{1}{\{45\rho + 50\}}\left[\frac{15}{36\rho^2 - 120\rho + 100}[18.864\rho + 216\rho + 14334.6\rho - 48319.17\rho + 34654.95\rho + 22612.5] - 75\rho + 2003\right] \_\_\_\_\_ (11)$$

By applying $\rho = 0.1, 0.2, \ldots\ldots\ldots, 1.0,$ in equation (11), we get different costs given in Table II. The corresponding curve is exhibited in Fig. 2.

TABLE II

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$T_c(N)$	116.7855	128.1064	145.4734	173.8312	220.4317	296.4726	420.0384	621.6926	915.8124	1525.4

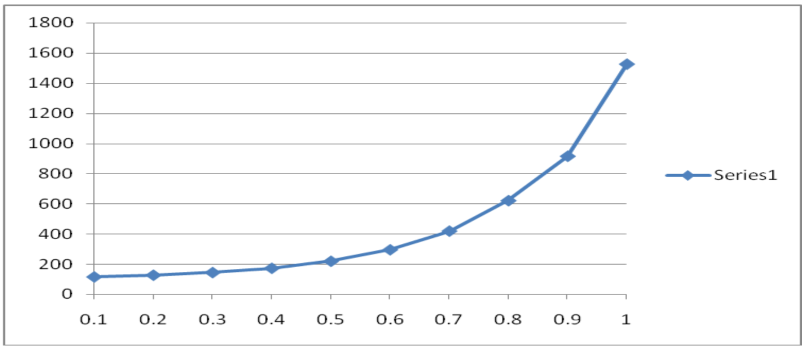

Fig. 2

C. $M^x/G/1$ queue with multiple vacations

The queueing system when the server becomes idle is not new. Miller (1964) was the first to study such a model, where the server is unavailable during some random length of time (referred to as vacation) for the M/G/1 queueing system. These generalizations are useful in model building in many real life situations such as digital communication, computer network and production, inventory systems [Takagi (1991) and Doshi (1986, 1990)].

The major general result for vacation models is the stochastic Decomposition result, which allows the system to be analysed by considering separately the distribution of the queue size with no vacation and the additional queue size due to vacation. This important result was first established by Fuhrmann and Cooper (1985) for generalized vacation as well as multiple vacati0on models, where the server keeps on taking a sequence of vacations of random length till it finds at least one unit in the system to start each busy period for the M/G/1 queueing systems.

Most studies are devoted to batch arrival queues with vacation because of its interdisciplinary character. The recent progress of $M^x/G/1$ type queueing models of this nature have been served by Chse and Lee (1995) and Medhi (1997).

Choudhury (2002) has discussed a queueing model $M^x/G/1$ in which customers arrive in batches according to a combined Poisson process. This queueing system is employed with multiple vacation policy denoted by M at both stationary and departure point of times.

The expected queue size at departure point of time is

$$L_q = \left(\lambda E(X)\right)^2 E(V)^2 + E\left(X(X-1)\right)\left[1 - V^*(\lambda)\right)^M E(V) + \left(1 - V^*(\lambda)\right)(\lambda)^M\right]$$
$$2\left[\lambda E(X)E(V)(1 - \left(V^*(\lambda)\right)^M + \left(1 - V^*(\lambda)\right)\left(V^*(\lambda)\right)^M E(x)\right] + P_+\lambda^2 E(B^2)E^2(X)$$
$$+ \lambda E(B)E\left(X(X-1)\right)\, 2(1-\rho) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (12)$$

Similarly the expected busy period and expected idle period are respectively given by

$$E(T) = \rho\, E(1)/(1-\rho)$$
$$= \left[1 - \left(V^*(\lambda)\right)^M\right]\rho\, E(V)/(1-\rho)[1 - V^*(\lambda)]$$
$$+ \left(V^*(\lambda)\right)^M E(X)E(B)/(1-\rho) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (13)$$

and

$$E(I) = \left[1 - \left(V^*(X)\right)^M\right]E(V)/[1 - V^*(\lambda)] + \left(V^*(X)\right)^M/\lambda \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (14)$$

These results reduce to the performances of $M^x/G/1$ with single vacation when M = 1.

The total expected cost function is stated as

$$T_c = C_h L_q + C_0\left[\frac{E(B)}{E(C)}\right] + (C_s + C_r)\left[\frac{1}{E(C)}\right] + C_a\left[\frac{E(I)}{E(C)}\right] \quad \text{\_\_\_\_\_\_\_\_\_\_} \quad (15)$$

Expressions for expected queue length, expected busy period and expected idle period given in equations (12), (13) and (14) are substituted in equation (15) and get a required cost function. For fixing different parameters and provide values for $\rho = 0.1, 0.2, \dots\dots, 0.9$, the required costs are obtained corresponding to the number of vacation.

$$TC = [733139.4269 - 433130.76(.32)^M - 357(.32)^M\rho - 1032.38685\rho$$
$$- 399.84\rho^2 + 236.64(.32)^M\rho^2][26.656 - 15.776\,(.32)^M$$
$$-26.656\rho + 15.776(.32)^M]^{-1}$$
$$+ [[10.2 - 8.9505\rho + 1.2495(.32)^M\rho + 326.4(.32)^M$$
$$+ 1.666(.65)^M + 1.36(.35)^M - 1.666(.65)^M\rho$$
$$- 1.36(.35)^M\rho)]\, 0.0833\rho - 0.0833(.32)^M\rho$$
$$+ 21.76(.32)^M + 0.0833(.65)^M + 0.068(.35)^M$$
$$- 0.0833(.65)^M\rho - 0.068(.35)^M\rho]^{-1}$$

TABLE III

M \ ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	30582.64	34874.03	39573.23	46008.4	55098.68	68784.36	91631.71	137361.3	274595.8
2	458486.9	523055.9	593544	690071.7	826426	1031711	1374422	2060366	4118884
3	6877187	7845785	8903107	10351022	12396336	15475616	20616270	30905440	61783202
4	1.03E+08	1.18E+08	1.34E+08	1.55E+08	1.86E+08	2.32E+08	3.09E+08	4.64E+08	9.27E+08
5	1.55E+09	1.77E+09	2E+09	2.33E+09	2.79E+09	3.48E+09	4.64E+09	6.95E+09	1.39E+10

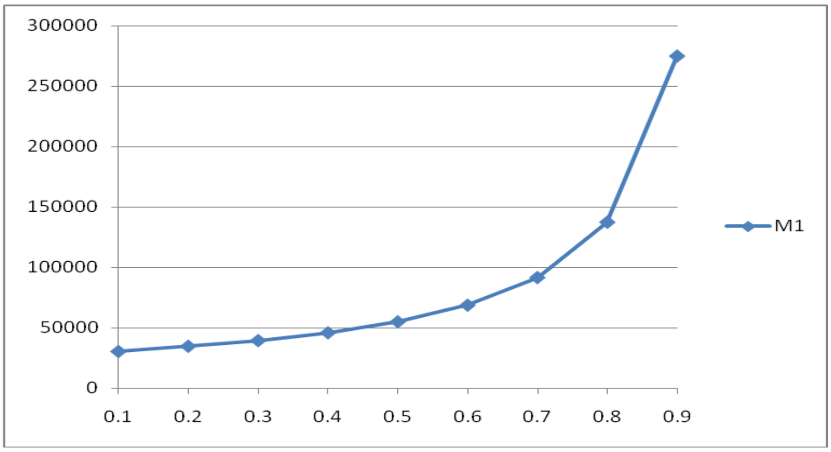

Fig. 3.1 Single vacation (M=1)

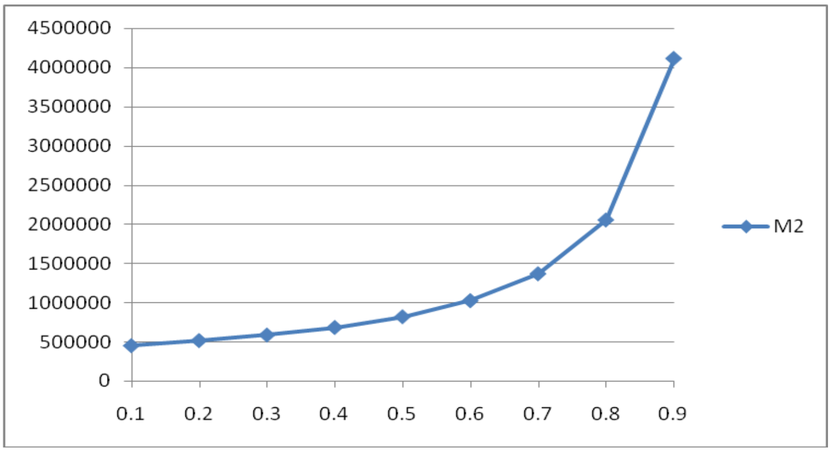

Fig. 3.2 Double vacations (M=2)

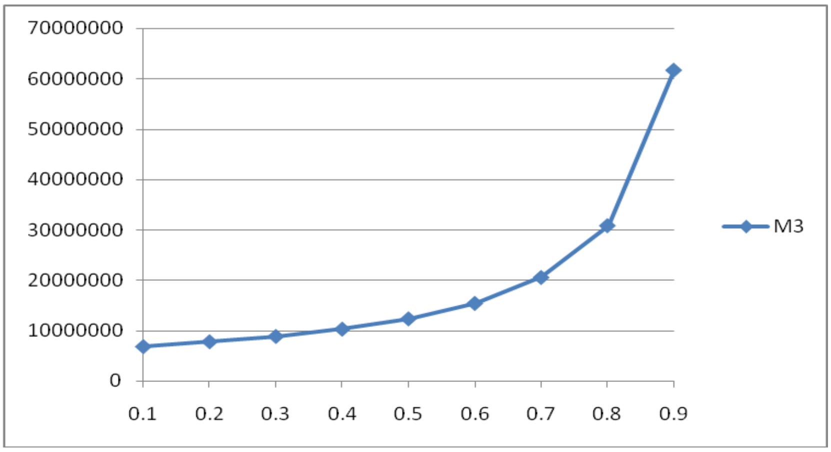

Fig. 3.3 Triple vacation (M=3)

D. M/E$_k$/1 queue with state dependent arrival rates

Jain et al (2011) have discussed M/E$_k$/1 queueing system with vacation policy and optional technique in repairs. In this system arrivals follow Poisson distribution with state dependent rates $\lambda_j, j = 0,1,2, \dots \dots \dots$ customers are served according to Erlang distribution with k identical and independent phases each with mean $^1/_{k\mu}$ and $^1/_{k\mu_1}$ while the server is in working vacation and busy states respectively. The vacation times of the server follows the exponential distribution with mean $\frac{1}{\eta}$. During the busy state; break downs occur and that follow Poisson distribution with break down rate $\propto$ and the repair rate is considered as β. After the break down, the system is presented to repair in which the repair man provide the first essential repair with rate β. The server may opt optional repair with probability σ or may join the system with probability $\bar{\sigma} = (1-\sigma)$.

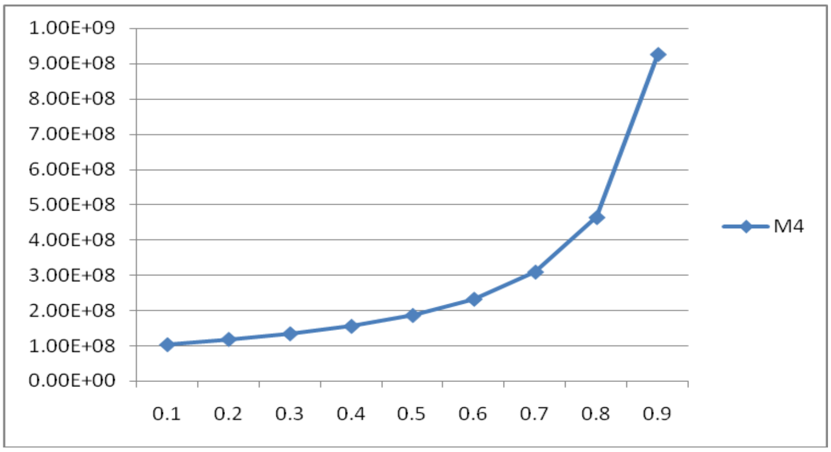

Fig. 3.4 Four vacations (M=4)

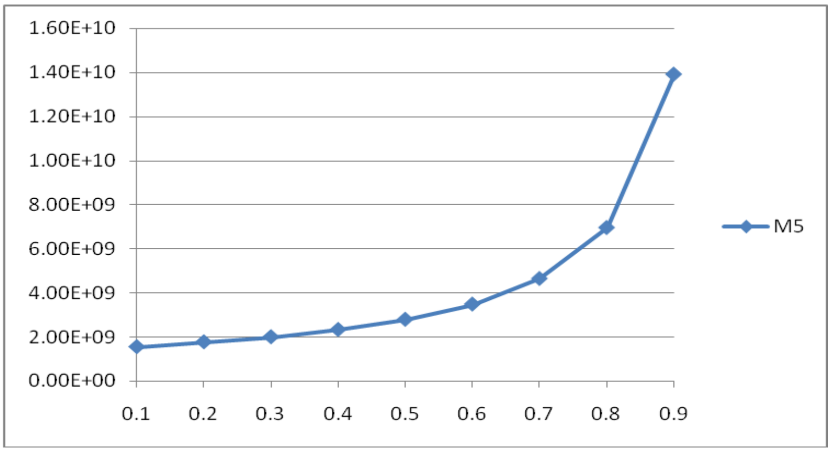

Fig. 3.5 Five vacations (M=5)

The steady state probabilities of the system are defined as $P_{0,0,V}, P_{n,i,V}, P_{n,i,B}, P_{n,i,D}, P_{n,i,Rj}$.

Based on the above probabilities, [Jain et al. (2011) have formulated the steady state equations and derived following probabilities.

$$P_r = \frac{b_0(1-z^*)P_{0,0,V}}{C_0} \quad \_\_\_\_\_\_\_\_ \quad (17)$$

$$P_B = \frac{b_0}{C_0}(1-z^*)\,\frac{[(\lambda_0+2\eta)-(1+C_0)^k(\lambda_0+\eta)]}{\mu_1(1+kV'(1))\left(1-(1+(0))^k\right)}\; C_0, P_{0,0,V} \quad \_\_\_\_\_\_\_\_ \quad (18)$$

$$P_D = \frac{\propto b_0}{\beta\; C_o}(1-z^*)\,\frac{[(\lambda_0+2\eta)-(1+C_0)^k(\lambda_0+\eta)]}{\mu_1(1+kV'(1))\left(1-(1+(0))^k\right)}\; P_{0,0,V} \quad \_\_\_\_\_\_\_\_ \quad (19)$$

$$P_{R_i} = \sigma\,\frac{\propto b_0}{\beta\; C_0}\;(1-z^*)\,\frac{[(\lambda_0+2\eta)-(1+C_0)^k(\lambda_0+\eta)]}{\mu_1(1+kV'(1))\left(1-(1+(0))^k\right)}\; P_{0,0,V} \quad \_\_\_\_\_\_\_\_ \quad (20)$$

$$P_{Rj} = \sigma\,\frac{\propto b_0}{\beta\; C_0}(1-z^*)\,\frac{[(\lambda_0+2\eta)-(1+C_0)^k(\lambda_0+\eta)]}{\mu_1(1+kV'(1))\left(1-(1+(0))^k\right)}\; P_{0,0,V}\; 2 \le j \le l \_\_\_ \quad (21)$$

The expected number of customers in the system is given by

$$EN_s = EN_V + EN_B + EN_D + \sum_{i=1}^{l} EN_{Ri} \quad \_\_\_\_\_\_\_\_ \quad (22)$$

where

$$EN_V = b_0(z^* - 1)\left[\frac{-(2C_0 + b_0) + (1 + C_0)^k (b_0 + C_0)}{1 - (1 + C_0)^k {C_0}^2}\right] P_{0,0}V \quad \_\_\_\_\_\_ \quad (23)$$

$$EN_B = b_0(1 - z)^{*} \begin{matrix} [\{k_\mu a^2 b^2 (4c^1 d^1 g - 6h(c'd'' + c''d') - 6e'(2c'''d' + 3c''d'' + 2c''d'''))\} \\ +2\eta\{2abc'd'(9fh + 2f'(4e' + h) + 3f''e' + fe''') - (15fe' + 4f'e + 2fe''')\Delta\}] \end{matrix} P_{0,0,V})$$

$$P_{0,0,V} = \frac{1}{\frac{b_0}{c_0}(1 - z^*) \times \left[1 + \frac{[(\lambda_0 + 2\eta) - (1 + c_0)^k (\lambda_0 + \eta)]}{\mu_1\left(1 + kV'(1)\right)(1 - (1 + c_0)^k)} \times \left(1 + \frac{\alpha}{\beta} + \frac{\alpha}{\beta}\Sigma_{j=1}^{1}\sigma^j\right)\right]} \quad \_\_\_\_\_\_\_\_ \quad (24)$$

$$EN_D = \frac{\propto}{\beta} P'_B(1) + \frac{\propto \lambda_2}{\beta^2} P_B(1) \quad \_\_\_\_\_\_\_\_ \quad (25)$$

$$EN_{RI} = \sigma \frac{\propto}{\beta} P'_B(1) + \frac{\sigma}{\bar{\sigma}} \frac{\propto \lambda_3 P_B(1)}{\beta^2} + \sigma \lambda_2 \frac{\propto}{\beta^2} P_B(1) \quad \_\_\_\_\_\_\_\_ \quad (26)$$

$$EN_{Rj} = \sigma^j \frac{\propto}{\beta} P'_B(1) + \frac{\sigma^j}{\bar{\sigma}} \frac{\propto}{\beta^2} \sum_{m=1}^{j} \lambda_{m+2} P_B(1) + \sigma^j \lambda_2 P_B(1), \quad 2 \le j \le l \quad \_\_\_\_ \quad (27)$$

$$EC = \frac{\frac{1}{\eta} + \frac{1}{\lambda_0\left(1 - B_V{}^*(\eta)\right)}}{\frac{b_0(1 - z^*)}{C_0} P_{0,0,V}} \quad \_\_\_\_\_\_\_\_ \quad (28)$$

For estimating the expected cost per unit time, the following costs are used.

C_0 - Start up cost per unit time for turning the server on
C_F - Shut down cost per unit time for turning the server off.
C_V - Cost incurred per unit time for keeping the server on working vacation.
C_B - Cost incurred per unit time for keeping the server busy.
C_D - Cost per unit time incurred on a failed server
C_{Rj} - Cost incurred per unit time for providing the j^{th} phase repair
C_H - holding cost per unit time per customer present in the system.

By using the above stated probabilities, the expected length of busy cycle, and expected number of customers in the system, the expected total cost per unit time is framed as

$$ETC = (C_0 + C_F)\frac{1}{EC} + C_v P_v + C_B P_B + C_D P_D + \sum_{j=1}^{l} C_{R_j} P_{R_j} + C_H EN_S \quad \_\_\_\_\_\_ \quad (29)$$

Substituting the expressions from (17) to (28) in the cost equation (29) and assuming the unknown cost parameters by numerical values.

The total costs are computed for varying λ and are given in table.

TABLE IV

λ	0.5	0.6	0.7	0.8	0.9	1
ETC	24.52	38.27	62.73	103.94	167.88	260.6

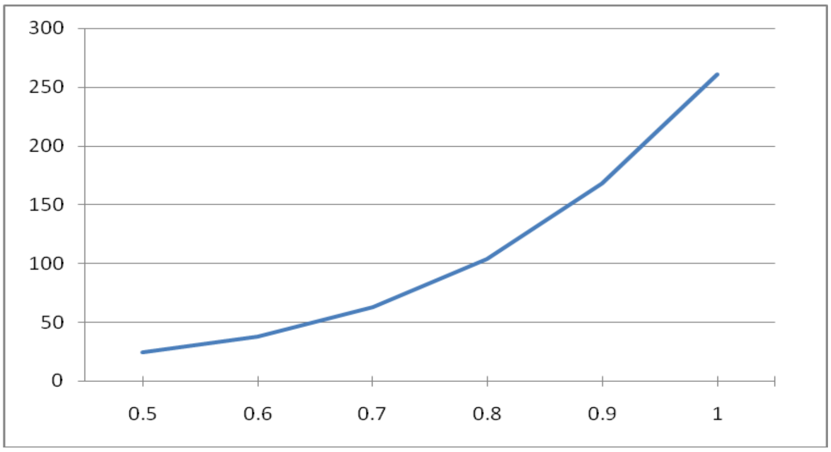

Fig. 4

III. CONCLUSION

The cost functions of Markovian arrival queues are analysed with various service conditions. The cost functions are framed by using expected queue length, expected busy period, expected idle period and etc. By the assumptions of various cost constraints, the expected total costs are computed for different values of ρ.

On seeing the curves (Fig (1) and Fig.(2)) of the models $M/E_k/1$ with single and bulk services, it is noted that the cost increases as ρ increases in both models. The cost of bulk service is less than the cost of single service for $\rho \leq 0.5$. But it is reversed for $\rho > 0.5$.

The costs of services for $M^x/G/1$ model with multiple vacation policy are computed and their curves (Figs.(3.1) to (3.5)) are drawn. The cost increases as ρ increases. Similarly, by considering the vacations, for increasing number of vacations, the costs are increasing.

On comparing the five figures, the cost of the service system with single vacation is lesser than that with multiple vacations. It is remarked that a queueing system with single vacation is better than the queue with multiple vacations.

On observing the model $M/E_k/1$ with breakdown, repair and vacation rules, the costs are much lower than the costs of other above mentioned models. Among the four cases, the model mentioned in section (2.4) is better to apply medical field. Suppose one may follow these conditions for queues with retrial and other characters, the researcher may surely get the cost functions and cost curves.

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